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Exact Analytical Characterization of Physical Traveling-Wave Structures in the Lonngren Equation

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ABSTRACT: In this study, we investigate the Lonngren wave equation by means of the generalized the $(m + 1/G')$-expansion method to obtain exact traveling-wave solutions for a variety of parameter regimes. This technique produces fresh analytical expressions that reveal the equation's dynamic behavior. We verify the correctness of these solutions by substituting them back into the original equation. In addition, three-dimensional surface plots and contour diagrams are presented to illustrate the physical characteristics of the resulting waveforms. Overall, our results demonstrate the effectiveness of the generalized expansion approach for solving nonlinear partial differential equations and deepen the theoretical understanding of wave propagation in applied settings.

KEYWORDS: Analytical solution, Generalized expansion method, Lonngren wave equation, Travelling wave solution.

INTRODUCTION

Nonlinear wave equations arise in diverse physical systems such as fluid dynamics, optical fibers, and plasma physics, where they often support localized traveling-wave solutions like solitons [1,2]. Over the years, various analytical techniques have been developed to construct these solutions. For instance, the tanh-method introduced by [2,3] provides a systematic procedure for deriving exact travelling-wave solutions in polynomial form, while the (G'/G)expansion method proposed by [4,5,6] yields a broad class of rational, hyperbolic, and trigonometric solutions by transforming a nonlinear equation into a second-order linear ordinary differential equation. Despite these advances, certain nonlinear models continue to reveal richer solution spectra that challenge classical techniques.

A prominent example is the Lonngren wave equation, which models voltage pulse propagation in nonlinear electrical transmission lines incorporating tunnel-diode elements [7,8]. They demonstrated experimentally that solitary waves can form in a discrete nonlinear transmission line, observing a clear transition from linear dispersion to stable soliton propagation as the input amplitude increased.

More recently, th $(m + 1/G')$e-expansion method has emerged as a powerful extension of the (G'/G) −expansion technique, introducing an additional parameter m and reciprocal derivative terms to enrich the solution approach [9,10,11]. This generalization systematically uncovers solution branches that may be inaccessible to classical methods without imposing integrability conditions. They successfully applied the $(m + 1/G')$ −expansion to the given equation, obtaining multiple new optical soliton solutions and demonstrating the method's algorithmic simplicity and broad applicability .

In this work, we apply the $(m + 1/G')$-expansion method to the Lonngren wave equation [12].

$$(u_{xx} - \alpha u + \beta u^2)_{tt} + u_{xx} = 0 \quad (1)$$

The Lonngren wave equation is commonly employed in areas such as telecommunication and network systems engineering. By reducing the original partial differential equation to an ordinary differential equation via a travelling-wave transformation and then employing the extended expansion approach, we derive new exact travelling-wave solutions in closed form. These solutions include bright-type solitons, kink-type structures, and modulated wave packets, each parameterized by integration constants and characterized by distinct physical behaviors. We discuss the relation between solution parameters and the properties of the electrical pulses in nonlinear transmission lines, thereby enriching the theoretical understanding of the Lonngren model and illustrating the effectiveness of the \((m + 1/G')\)-expansion method for solving nonlinear wave equations.

The structure of this paper is organized as follows: In Section 2, the general framework of the $(m + 1/G')$-expansion method is introduced, and the algorithmic steps of the solution approach are outlined. Section 3 applies this method to the Lonngren wave equation, yielding exact travelling wave solutions through analytical procedures. In Section 4, graphical representations including 3D and contour plots are provided for each solution, offering visual insight into the waveforms' structure and behavior. Section 5 presents

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a physical interpretation of the obtained solutions, discussing their significance in nonlinear wave dynamics. Finally, Section 6 concludes the study by summarizing the findings and suggesting possible directions for future research.

GENERAL FORM OF THE $(m + 1/G')$ −EXPANSION METHOD

Consider the nonlinear partial differential equation as the following general form [13,14],

$$P(u, u_x, u_t, u_{xx}, u_{tt}, u_{xt}, \dots) = 0 \tag{2}$$

where $P$is a polynomial and $u = u(x,t)$.

Then suppose the wave variables takes the form $\xi = x - kt$. Eq.(2) can be transformed into the nonlinear ODE,

$$N(U, U', U'', \dots) = 0 \tag{3}$$

where prime denotes the derivative with respect to ξ.

Step1

Suppose that the solution of Eq.(2) can be written as a finite power series with the form

$$U(\xi) = \sum_{i=-n}^{n} a_i (m + F)^i \tag{4}$$

Where a_0, a_i $(i = \mp 1, \dots \mp n)$ and $m$are constants. The degree of the power series is determined by considering the homogeneous balance between nonlinear term in Eq.(2) and the highest order derivative $F = \frac{1}{G'}$ where $G(\xi)$ satisfies

$$G'' + (\lambda + 2m\mu)G' + \mu = 0 \tag{5}$$

Step 2

Substitute the solution of Eq.(3) into Eq.(4) and use Eq.(5), then collect all terms with the same order of the $(m + F)^i$ to obtain the system of algebraic equations for U, a_0, a_i $(i = \mp 1, \dots, \mp n), \lambda, \mu$.

Step 3

Solve the obtained system and substitute U, a_0, a_i $(i = \mp 1, \dots, \mp n)$ and the general solution of the LODE Eq.(5) in to Eq.(3) to get the exact solution of Eq. 2.

Remark The solution of the LODE $G'' + (\lambda + 2m\mu)G' + \mu = 0$ is

$$G = -\frac{\mu\xi}{\lambda + 2\mu m} - \frac{A_1}{\lambda + 2\mu m} e^{-(\lambda + 2\mu m)\xi} + A_2, \tag{6}$$

where A_1, A_2 are constants depending on given boundary conditions. Thus we have

$$\frac{1}{G'} = \frac{\lambda + 2\mu m}{-\mu + (\lambda + 2\mu m)A_1\left(\cosh((\lambda + 2\mu m)\xi) - \sinh((\lambda + 2\mu m)\xi)\right)}, \tag{7}$$

$$\left(\frac{1}{G'}\right)' = \mu\left(m + \frac{1}{G'}\right)^2 + \lambda\left(m + \frac{1}{G'}\right) - m(\lambda + m\mu).$$

APPLICATIONS OF THE $(m + 1/G')$ −EXPANSION METHOD

In this section we use the $\left(m + \frac{1}{G'}\right)$-expansion method to obtain the travelling wave solutions for Lonngren-wave equation. To apply the $\left(m + \frac{1}{G'}\right)$ −expansion method, we begin with the following transformation.

$$u(x,t) = U(\xi), \ \xi = k(x - ct) \tag{8}$$

Substituting Eq.(8) into Eq.(1) we have,

$$c^2k^2U'' + (1 - \alpha c^2)U + \beta c^2 U^2 = 0 \tag{9}$$

where $U' = \frac{dU}{d\xi}, U'' = \frac{d^2U}{d\xi^2}$.

By taking the balance we obtain $n = 2$. When we enter the value of balance into Eq.(9), we get

$$U(\xi) = a_{-2}(m + F)^{-2} + a_{-1}(m + F)^{-1} + a_0 + a_1(m + F) + a_2(m + F)^2 \tag{10}$$

Following Step 2 in the algorithm of method yields the following algebraic system,

$$\left(m + \frac{1}{G'}\right)^{-4} : 6c^2k^2m^2(\lambda + m\mu)^2 a_{-2} + c^2\beta a_{-2}^2 = 0$$

$$\left(m + \frac{1}{G'}\right)^{-3} : -10c^2k^2m\lambda(\lambda + m\mu)a_{-2} + 2c^2k^2m^2(\lambda + m\mu)^2 a_{-1} + 2c^2\beta a_{-2}a_{-1} = 0$$

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$\left(m+\frac{1}{G'}\right)^{-2}: (1-c^2\alpha)a_{-2}+4c^2k^2\lambda^2a_{-2}-8c^2k^2m\mu(\lambda+m\mu)a_{-2}-3c^2k^2m\lambda(\lambda+m\mu)a_{-1}+c^2\beta a_{-1}^2+2c^2\beta a_{-2}a_0=0$

$\left(m+\frac{1}{G'}\right)^{-1}: 6c^2k^2\lambda\mu a_{-2}+(1-c^2\alpha)a_{-1}+c^2k^2\lambda^2a_{-1}-2c^2k^2m\mu(\lambda+m\mu)a_{-1}+2c^2\beta a_{-1}a_0+2c^2\beta a_{-2}a_1=0$

$\left(m+\frac{1}{G'}\right)^{0}: 2c^2k^2\mu^2a_{-2}+c^2k^2\lambda\mu a_{-1}+(1-c^2\alpha)a_0+c^2\beta a_0^2-c^2k^2m\lambda(\lambda+m\mu)a_1+2c^2\beta a_{-1}a_1+2c^2k^2m^2(\lambda+m\mu)^2a_2$
$+2c^2\beta a_{-2}a_2=0$

$\left(m+\frac{1}{G'}\right)^{1}: (1-c^2\alpha)a_1+c^2k^2\lambda^2a_1-2c^2k^2m\mu(\lambda+m\mu)a_1+2c^2\beta a_0a_1-6c^2k^2m\lambda(\lambda+m\mu)a_2+2c^2\beta a_{-1}a_2=0$

$\left(m+\frac{1}{G'}\right)^{2}: 3c^2k^2\lambda\mu a_1+c^2\beta a_1^2+(1-c^2\alpha)a_2+4c^2k^2\lambda^2a_2-8c^2k^2m\mu(\lambda+m\mu)a_2+2c^2\beta a_0a_2=0$

$\left(m+\frac{1}{G'}\right)^{3}: 2c^2k^2\mu^2a_1+10c^2k^2\lambda\mu a_2+2c^2\beta a_1a_2=0$

$\left(m+\frac{1}{G'}\right)^{4}: 6c^2k^2\mu^2a_2+c^2\beta a_2^2=0$

Case 1:

When, $a_{-2}=0; a_{-1}=0; a_0=0; a_1=\frac{6(-1+c^2\alpha)}{c^2m\beta}; a_2=\frac{6-6c^2\alpha}{c^2m^2\beta}; \lambda=-m\mu; k=-\frac{\sqrt{-1+c^2\alpha}}{cm\mu}$

One could gain the following solution,

$$u(x,t)=\frac{6\mathrm{A1}e^{\frac{(-ct+x)\sqrt{-1+c^2\alpha}}{c}}m(-1+c^2\alpha)}{c^2(-1+\mathrm{A1}e^{\frac{(-ct+x)\sqrt{-1+c^2\alpha}}{c}}m)^2\beta}.$$

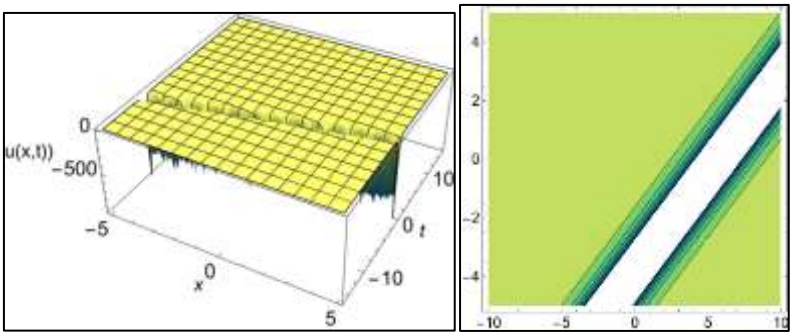

Figure1: 3D and contour graph of travelling wave solution when $m=0.01;\ \mathrm{A1}=0.7; c=1.5; \alpha=1.2; \beta=0.3$

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Case2:

When , $a_{-2}=((9/2-6i)m^2(-1+c^2\alpha))/(c^2\beta)$; $a_{-1}=-(((3-9i)m(-1+c^2\alpha))/(c^2\beta))$;
$a_0=((1/2+3i)(1-c^2\alpha))/(c^2\beta)$; $a_1=0; a_2=0; \lambda=(-2-2i)m\mu$;
$k=-\left(\left(\sqrt{-1+c^2\alpha}\right)/(2cm\mu)\right)$ we can obtain the following soliton solution,

$$u(x,t)=\frac{i((4+3i)e^{\frac{2i(-ct+x)\sqrt{-1+c^2\alpha}}{c}}-(8+16i)\mathrm{A1}e^{\frac{i(-ct+x)\sqrt{-1+c^2\alpha}}{c}}m+4i\mathrm{A1}^2m^2)(-1+c^2\alpha)}{c^2((1+2i)e^{\frac{i(-ct+x)\sqrt{-1+c^2\alpha}}{c}}+2i\mathrm{A1}m)^2\beta}.$$

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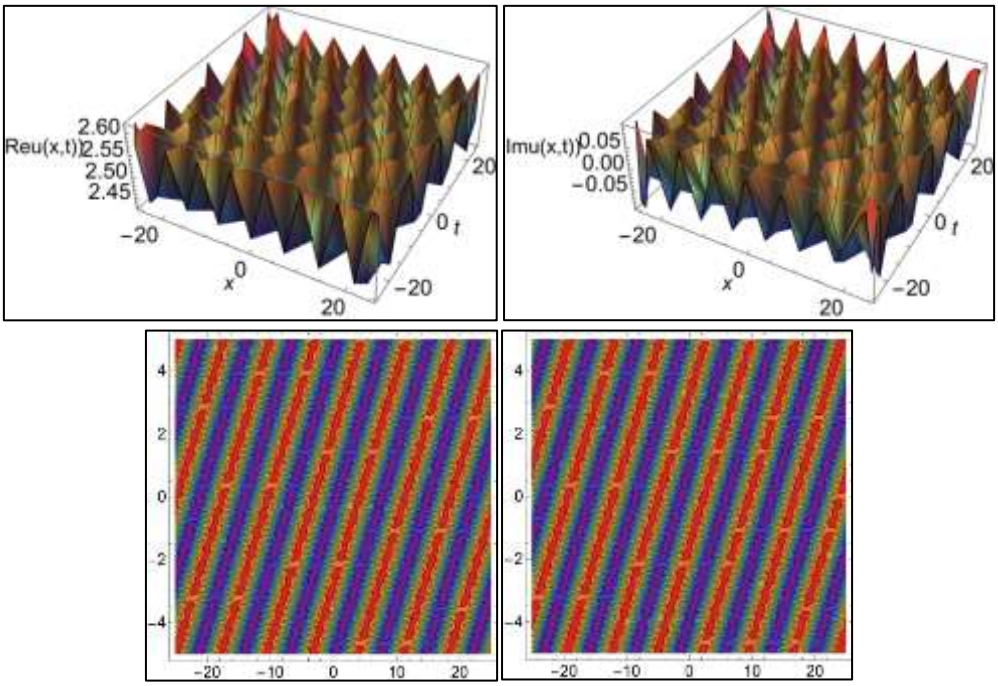


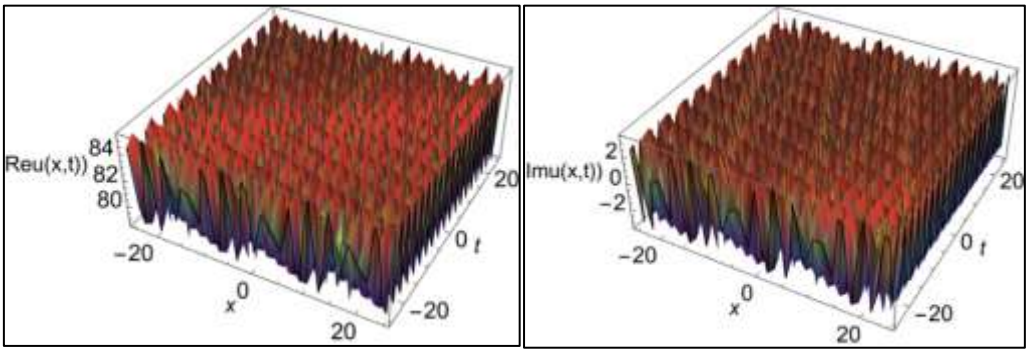
Figure 2: 3D and contour graph of travelling wave solution when $m = 0.01; A1 = 0.7; c = 1.5; \alpha = 1.2; \beta = 0.3$

Case3:

When, $a_{-2} = 0; a_{-1} = 0; a_0 = \frac{-1+c^2\alpha}{c^2\beta}; a_1 = \frac{6-6c^2\alpha}{c^2 m\beta}; a_2 = \frac{6(-1+c^2\alpha)}{c^2 m^2\beta}; \lambda = -m\mu;$

$k = -\frac{i\sqrt{-1+c^2\alpha}}{cm\mu}$ we can obtain the following soliton solution,

$$u(x,t) = \frac{(1+\mathrm{A1}e^{\frac{i(-ct+x)\sqrt{-1+c^2\alpha}}{c}}m(4+\mathrm{A1}e^{\frac{i(-ct+x)\sqrt{-1+c^2\alpha}}{c}}m))(-1+c^2\alpha)}{c^2(-1+\mathrm{A1}e^{\frac{i(-ct+x)\sqrt{-1+c^2\alpha}}{c}}m)^2\beta}$$



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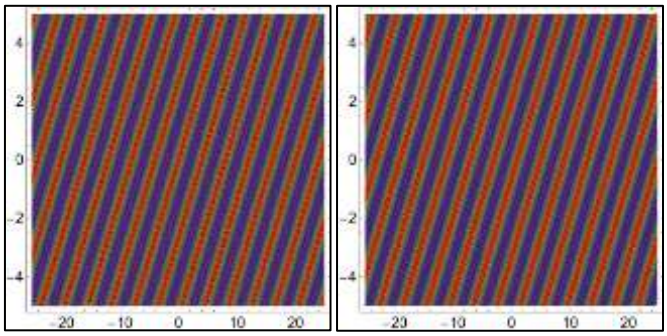

Figure 3: 3D and contour graph of travelling wave solution when $m = 0.01; A1 = 0.7; c = 1.5; \alpha = 2.9; \beta = 0.03$.

GRAPHICAL INTERPRETATION OF THE SOLUTIONS

The graphical representations of the obtained travelling wave solutions provide valuable insight into the physical behaviors and structural characteristics of the nonlinear waveforms governed by the Lonngren equation. Each figure corresponds to a specific case solution derived through the $(m + 1/G')$-expansion method and reflects a distinct type of nonlinear wave propagation. Figure 1 illustrates the travelling wave solution obtained in Case 1, which presents a real-valued bright soliton profile. The 3D plot reveals a single, localized hump that maintains its form while travelling, a classic indicator of bright solitary waves. These structures emerge when nonlinearity and dispersion are in balance, resulting in energy localization. The accompanying contour plot displays circular symmetry centered around the wave peak, further confirming the localized nature and the solitary structure of the solution. Figure 2 corresponds to Case 2 and showcases a complex-valued modulated wave solution. Both the real and imaginary parts of the solution are plotted, each demonstrating oscillatory behavior in space. This structure resembles a dispersive wave packet, where modulation effects are evident. The real and imaginary components are out of phase, indicating internal wave interactions and possible phase shifts. The contour plots for both components reveal interference-like patterns, highlighting the wave's multi-modal character and complex evolution. Figure 3, derived from Case 3, represents a kink-type or dark soliton structure. In this solution, the wave connects two asymptotically distinct states, which is characteristic of domain wall propagation or phase transitions in bistable media. The real part transitions smoothly from one level to another, while the imaginary part adds further depth to the phase dynamics. The 3D and contour plots emphasize this transition and demonstrate the wave's spatial extent, highlighting its topological features and propagation stability.

PHYSICAL INTERPRETATION

The travelling wave solutions obtained for the Lonngren equation reveal rich nonlinear dynamics that reflect physically significant behaviors observed in dispersive media. The first solution represents a bright-type solitary wave, characterized by a localized, symmetric peak that propagates stably. This form typically appears when nonlinearity and dispersion are in balance, leading to energy confinement in space. The second solution is complex-valued and exhibits both real and imaginary oscillatory components, indicating modulated waveforms or wave packets. These solutions suggest internal phase dynamics and interference effects, commonly seen in nonlinear optics and dispersive plasma environments. The third solution displays a kink-type or dark soliton profile, connecting two distinct asymptotic states. Such structures are associated with phase transitions or domain wall propagation, indicating bistable dynamics in the system. Together, these solutions highlight the versatility of the Lonngren equation in modeling various nonlinear wave phenomena, with their structure and behavior being highly dependent on physical parameters such as nonlinearity, dispersion, and wave velocity.

CONCLUSION

In this study, we have successfully derived travelling wave solutions of the nonlinear Lonngren equation using the $(m + 1/G')$-expansion method. This analytical technique allowed us to transform the original nonlinear partial differential equation into an ordinary differential equation and obtain explicit solutions in closed form. The results include bright-type solitary waves, complex-valued modulated waveforms, and kink-type structures, each representing physically significant behaviors such as energy concentration, oscillatory dispersion, and phase transition fronts in nonlinear media.Graphical analysis of the solutions, including

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3D and contour representations, provided further insight into the nature and stability of the waveforms. These visualizations confirmed that the solutions exhibit distinct nonlinear features depending on the model parameters, highlighting the versatility of the Lonngren equation in describing various nonlinear wave phenomena in plasma dynamics, nonlinear optics, and related fields. This research may be extended in several directions. One potential path is to investigate the stability properties of the obtained solutions through numerical simulations or spectral analysis. Another avenue is the application of the $(m + 1/G')$ −expansion method to more generalized or perturbed versions of the Lonngren equation, such as those involving variable coefficients or higher-dimensional terms. Additionally, the interaction dynamics between multiple solitary waves, as well as the effects of external forces or damping terms, could be explored to enhance the physical applicability of the model. These extensions would not only broaden the understanding of the equation's solution space but also provide practical relevance in modeling real-world nonlinear wave processes.

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