



A Dependable Numerical Approach for Solving a Nonlinear Generalized Fractional Distributed-Order Black-Scholes Equation

Noorulhaq Ahmadi¹, Sayed Lamsoon Sayedi², Hameed Gul Ahmadzai³

^{1,2,3}Department of Mathematics, Faculty of Education, Paktia University, Paktia, AFGHANISTAN

ABSTRACT: In this research, we explore the nonlinear generalized distributed-order time-fractional Black-Scholes equation using an implicit numerical approach. Finite difference techniques are employed to approximate the time and spatial derivatives. Our numerical results exhibit high accuracy, underscoring the method's robustness in addressing financial models. Additionally, our approach offers significant advantages in computational efficiency and stability. By using the implicit method, we ensure solution stability even with larger time steps, which is particularly beneficial for long-term financial modeling. The implications of this study extend beyond financial engineering. The methodologies developed can be adapted to solve various fractional differential equations in different scientific and engineering fields. The successful application of these techniques to the Black-Scholes equation suggests their potential utility in other areas requiring precise and efficient numerical solutions.

KEYWORDS: AMS, Black-Scholes model, Generalized Fractional distributed-order derivatives, Trust region-dogleg algorithm.

Subject Classification: 26A33; 35G25; 65D07.

1. INTRODUCTION

The Black-Scholes equation, also known as the Black-Scholes-Merton model, is a fundamental concept in financial mathematics, particularly in the field of options pricing. Developed by Fischer Black, Myron Scholes, and Robert Merton in 1973, this model provides a theoretical estimate of the price of European style options [1,2].

The Black-Scholes model assumes that the price of the underlying asset follows a geometric Brownian motion with constant drift and volatility. The key variables in the model are as follows:

S : Current price of the underlying asset,
 K : Strike price of the option,
 T : Time to maturity of the option,
 r : Risk-free interest rate,
 σ : Volatility of the underlying asset.

The model leads to a partial differential equation (PDE) that describes the price of the option over time. The Black-Scholes PDE is given by:

$$\frac{\partial u}{\partial t} + rS \frac{\partial u}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - ru = 0$$

where u is the price of the option as a function of the underlying asset price S and time t .

The derivation of the Black-Scholes equation involves several steps, primarily using Ito's Lemma and the concept of a risk-neutral portfolio. Here's a brief outline:

1. Modeling the Asset Price: Assume the asset price follows a stochastic differential equation (SDE):

$$dS = \mu S dt + \sigma S dW$$

where μ is the drift rate, σ is the volatility, and dW is a Wiener process.

2. Applying Ito's Lemma: For a function $u(S, t)$, Ito's Lemma gives:

$$du = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial S} dS + \frac{1}{2} \frac{\partial^2 u}{\partial S^2} (dS)^2$$



3. Constructing a Risk-Free Portfolio: Create a portfolio consisting of the option and a quantity Δ of the underlying asset such that the portfolio is risk-free. This leads to:

$$d\Pi = dC - \Delta dS$$

4. Eliminating Risk: By choosing $\Delta = \frac{\partial u}{\partial S}$, the portfolio becomes risk-free, and its return must equal the risk-free rate r :

$$d\Pi = \left(\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} \right) dt$$

5. Equating to Risk-Free Return: Set the return of the risk-free portfolio equal to $r\Pi$:

$$\frac{\partial u}{\partial t} + rS \frac{\partial u}{\partial S} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 u}{\partial S^2} - ru = 0$$

The solution to the Black-Scholes PDE for a European call option is given by the Black-Scholes formula:

$$u(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

where

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}}$$

$$d_2 = d_1 - \sigma\sqrt{T - t}$$

and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution.

The Black-Scholes model is widely used for pricing European options and has been extended to other types of derivatives. However, it has limitations, such as assuming constant volatility and interest rates, and not accounting for dividends. Despite these limitations, it remains a cornerstone of modern financial theory (see [3, 4]).

The classical Black-Scholes equation assumes constant volatility and risk-free interest rates, which can be limiting in real-world scenarios. To address these limitations, researchers have introduced the fractional Black-Scholes equation (FBSE), which incorporates fractional derivatives to better capture market behaviors such as long-range dependence and heavy-tailed distributions. The fractional Black-Scholes equation modifies the classical model by replacing the standard derivative with a fractional derivative. This allows the model to account for the memory and hereditary properties of financial markets. The general form of the FBSE for a European call option can be written as:

$$\frac{\partial u}{\partial t} + \frac{1}{2} \sigma^2 S^2 \frac{\partial^\alpha u}{\partial S^\alpha} + rS \frac{\partial u}{\partial S} - ru = 0$$

Here, $\frac{\partial^\alpha}{\partial S^\alpha}$ represents the fractional derivative of order α , u is the option price, S is the underlying asset price, σ is the volatility, and r is the risk-free interest rate. Many numerical methods have been developed to numerically solve a variety of equations such as the Black-Scholes equation (see [5-9]).

The fractional Black-Scholes equation is particularly useful for modeling market anomalies such as fat tails and leptokurtic distributions of asset returns, which are not well captured by the classical Black Scholes model. By providing a more accurate representation of market behaviors, the FBSE aids in better risk assessment and management for financial institutions. It offers a more flexible framework for pricing European options, especially in markets exhibiting fractal characteristics. Additionally, various numerical methods, such as the fractional generalized homotopy analysis method, have been developed to solve the FBSE, providing efficient and accurate solutions for practical applications. For a detailed review and solution techniques of the fractional Black-Scholes equations, please see [10-12] and references therein. A linear generalized fractional distributed-order Black-Scholes equation was studied by Zhang et al. [13].

In this research, we explore a novel model known as the nonlinear generalized distributed-order time fractional Black-Scholes equation. This sophisticated model is specifically tailored to capture the dynamics of pricing European double barrier options. By incorporating the complexities of nonlinearities and distributed order fractional derivatives, it offers a more precise and adaptable framework for financial modeling. Utilizing this model, we aim to improve the accuracy of option pricing, especially in situations where traditional models are inadequate. The detailed formulation and implications of this equation are discussed as follows:

$$(\partial_t + \phi^\delta \partial_t^* - A \partial_{xx} + B \partial_x + C) \phi(x, t) = g(x, t), x \in (c_1, c_2), t \in (0, T], \quad (1)$$

with homogeneous Dirichlet boundary conditions and the initial condition



$$\phi(x, 0) = \phi_0(x), c_1 \leq x \leq c_2. \quad (2)$$

Here $\phi(x, t) = u(\log(S), T - t)$, $A = \frac{\sigma^2}{2}$, $B = A - r$, $C = r$, $g \in C[c_1, c_2] \times C[0, T]$ and $\phi_0 \in C^2[c_1, c_2]$ are considered as known functions. Here, ${}^\delta \partial_t^*$ denotes the generalized distributed-order differential operator which is defined as follow

$${}^\delta \partial_t^* \phi(x, t) = \int_{\tau_*}^{\tau^*} \delta(\tau, x, t) \partial_t^\tau \phi(x, t) d\tau, \quad (3)$$

where $\delta(\tau, x, t)$ is known density function and $\partial_t^\tau \phi(x, t) = \frac{1}{\Gamma(1-\tau)} \int_0^t \frac{1}{(t-s)^\tau} \frac{\partial \phi(x, s)}{\partial s} ds$.

The generalized fractional distributed-order Black-Scholes model has recently attracted considerable attention due to its capacity to extend previous models [13], [14]. Despite its promise, there have been few numerical solutions for this model. This article aims to fill this gap by providing a numerical solution.

The paper is structured as follows: Section 2 explores a method for solving Equation (1) using finite difference techniques. Section 3 presents two numerical examples, with results illustrated through figures to demonstrate the methods effectiveness. Finally, Section 4 concludes with a summary of our findings and recommendations for future research.

2. ANALYSIS OF THE METHOD

In this section, we first make a numerical scheme to solve problem (1)-(2) using certain approximations of finite differences. Then we will analyze the convergence analysis of the numerical method.

2.1 Solution Method

Let $x_l = c_1 + l\zeta$, $l = 0, 1, \dots, L$, $\zeta = \frac{c_2 - c_1}{L}$, $t_n = n\kappa$, $n = 0, 1, \dots, N$, $\kappa = \frac{T}{N}$, $t_{n+\frac{1}{2}} = t_n + \frac{1}{2}\kappa$ and $\tau_k = \tau_* + k\Delta\tau$, $k = 0, 1, \dots, K$, $\Delta\tau = \frac{\tau^* - \tau_*}{K}$. Also, assume that $\delta_k^n = \delta(\tau_k, x, t_n)$ and $\phi^n = \phi(x, t^n)$.

Using the forward approximation $\frac{\partial \phi(x, s)}{\partial s} = \frac{\phi^{i+1} - \phi^i}{\kappa} + O(\kappa)$, we have:

$$\begin{aligned} \partial_t^\tau \phi^n &= \frac{1}{\Gamma(1-\tau)} \int_0^{t_n} \frac{1}{(t-s)^\tau} \frac{\partial \phi(x, s)}{\partial s} ds \\ &= \frac{1}{\Gamma(1-\tau)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{(t-s)^\tau} \frac{\partial \phi(x, s)}{\partial s} ds \\ &= \frac{1}{\kappa \Gamma(1-\tau)} \sum_{i=0}^{n-1} \int_{t_i}^{t_{i+1}} \frac{1}{(t-s)^\tau} (\phi^{i+1} - \phi^i + O((\kappa)^2)) ds \\ &= \frac{1}{\kappa \Gamma(1-\tau)} \sum_{i=0}^{n-1} (\phi^{i+1} - \phi^i) W_{i,n}^\tau + O(\kappa) \end{aligned} \quad (4)$$

where $W_{i,n}^\tau = \int_{t_i}^{t_{i+1}} \frac{1}{(t-s)^\tau} ds$. Now for approximation ${}^\delta \partial_t^* \phi^n$ we use the trapezoidal integration method as follows

$${}^\delta \partial_t^* \phi^n = \frac{\Delta\tau}{2} \left(\delta_0^n \partial_t^{\tau_0} \phi^n + 2 \sum_{k=1}^{K-1} \delta_k^n \partial_t^{\tau_k} \phi^n + \delta_K^n \partial_t^{\tau_K} \phi^n \right) + O((\Delta\tau)^2) \quad (5)$$

By substituting (4) in (5) we obtain

$${}^\delta \partial_t^* \phi^n = \frac{\Delta\tau}{2\kappa} \sum_{i=0}^{n-1} (\phi^{i+1} - \phi^i) U_i^n + O((\Delta\tau)^2 + (\Delta\tau)\kappa) \quad (6)$$

where

$$U_i^n(x) = \frac{\delta_0^n(x) W_{i,n}^{\tau_0}}{\Gamma(1-\tau_0)} + 2 \sum_{k=1}^{K-1} \frac{\delta_k^n(x) W_{i,n}^{\tau_k}}{\Gamma(1-\tau_k)} + \frac{\delta_K^n(x) W_{i,n}^{\tau_K}}{\Gamma(1-\tau_K)}$$

By putting $t = t_{n+\frac{1}{2}}$ in (1) and using Crank-Nicolson method we get

$$\begin{aligned} & \phi^{n+1} - \phi^n + \frac{\kappa}{2} (\phi^{n+1\delta} \partial_t^* \phi^{n+1} + \phi^{n\delta} \partial_t^* \phi^n) - \frac{A\kappa}{2} (\phi_{xx}^{n+1} + \phi_{xx}^n) + \frac{B\kappa}{2} (\phi_x^{n+1} + \phi_x^n) \\ & + C\kappa + O(\kappa^2) = \kappa g \left(x, t_{n+\frac{1}{2}} \right) \end{aligned} \quad (7)$$

Substituting (6) in (7) get

$$\begin{aligned} & \left(1 + \frac{\Delta\tau}{4} \sum_{i=0}^n (\phi^{i+1} - \phi^i) U_i^{n+1} \right) \phi^{n+1} - \frac{A\kappa}{2} \phi_{xx}^{n+1} + \frac{B\kappa}{2} \phi_x^{n+1} \\ & = \phi^n - \phi^n \frac{\Delta\tau}{4} \sum_{i=0}^{n-1} (\phi^{i+1} - \phi^i) U_i^n + \frac{A\kappa}{2} \phi_{xx}^n - \frac{B\kappa}{2} \phi_x^n - C\kappa + \kappa g \left(x, t_{n+\frac{1}{2}} \right) \\ & + O((\Delta\tau)^2 \kappa + (\Delta\tau) \kappa^2 + \kappa^2) \end{aligned} \quad (8)$$

Using central finite difference approximations for spatial approximation in (8) we obtain

$$\Theta_l^0 \phi_l^1 + \frac{\Delta\tau}{4} U_0^1(x_l) (\phi_l^1)^2 - C_1 \phi_{l-1}^1 - C_2 \phi_{l+1}^1 = R_l^0 + \varepsilon$$

$$\Theta_l^n \phi_l^{n+1} + \frac{\Delta\tau}{4} U_n^{n+1}(x_l) (\phi_l^{n+1})^2 - C_1 \phi_{l-1}^{n+1} - C_2 \phi_{l+1}^{n+1} = R_l^n + \varepsilon, n > 0 \quad (9)$$

where $\phi_l^n = \phi(x_l, t_n)$, $\varepsilon = O((\Delta\tau)^2 \kappa + (\Delta\tau) \kappa^2 + \kappa^2 + \varsigma \kappa)$ and

$$\begin{aligned} \Theta_l^0 &= 1 - \frac{\Delta\tau}{4} U_0^1(x_l) \phi^0(x_l) + \frac{A\kappa}{\varsigma^2} \\ \Theta_l^n &= 1 + \frac{A\kappa}{\varsigma^2} - \frac{\Delta\tau}{4} U_n^{n+1}(x_l) \phi_l^n + \frac{\Delta\tau}{4} \sum_{i=0}^{n-1} (\phi_l^{i+1} - \phi_l^i) U_i^{n+1}(x_l), n > 0 \\ R_l^0 &= \phi^0(x_l) + \frac{A\kappa}{2} \phi_{xx}^0(x_l) - \frac{B\kappa}{2} \phi_x^0(x_l) - C\kappa + \kappa g \left(x_l, t_{\frac{1}{2}} \right) \\ R_l^n &= Y_l^n \phi_l^n + c_1 \phi_{l-1}^n + c_2 \phi_{l+1}^n - C\kappa + \kappa g \left(x_l, t_{n+\frac{1}{2}} \right), n > 0 \\ C_1 &= \frac{A\kappa}{2\varsigma^2} + \frac{B\kappa}{4\varsigma}, C_2 = \frac{A\kappa}{2\varsigma^2} - \frac{B\kappa}{4\varsigma} \\ Y_l^n &= 1 - \frac{\Delta\tau}{4} \sum_{i=0}^{n-1} (\phi_l^{i+1} - \phi_l^i) U_i^n(x_l) - \frac{A\kappa}{\varsigma^2} \end{aligned}$$

Therefore the numerical scheme where be obtained by elimination of error term ε in (9) as follows

$$\Theta_l^0 \Phi_l^1 + \frac{\Delta\tau}{4} U_0^1(x_l) (\Phi_l^1)^2 - c_1 \Phi_{l-1}^1 - c_2 \Phi_{l+1}^1 = R_l^0, l = 1, 2, \dots, L-1$$

$$\Theta_l^n \Phi_l^{n+1} + \frac{\Delta\tau}{4} U_n^{n+1}(x_l) (\Phi_l^{n+1})^2 - c_1 \Phi_{l-1}^{n+1} - c_2 \Phi_{l+1}^{n+1} = R_l^n, n > 0, l = 1, 2, \dots, L-1 \quad (10)$$

where Φ_l^n represents the approximation of ϕ_l^n .

Its noticeably that $\Phi_l^0, \phi^0(x_l), \phi_x^0(x_l)$, and $\phi_{xx}^0(x_l), l = 1, 2, \dots, L-1$ can be obtained from initial condition (2), and then $\Phi_l^n, l = 1, 2, \dots, L-1$ can be obtained using solution of system (10). The system of nonlinear equations (2) can be solved using the trust region-dogleg algorithm [15], [16].

2.2 Error Analysis

Theorem 1. The method used to solve the system of nonlinear equations (10) has first-order convergence.

Proof: To achieve first-order convergence of the numerical method for solving the system of nonlinear equations given in Equation (10), it is sufficient to demonstrate that the Hessian matrix of each equation in the system is bounded (refer to Section 3 in [16]). Given that the Hessian matrices for the equations in Equation (10) are zero, the infinity norm of all Hessian matrices associated with the system is consequently bounded.

Furthermore, this boundedness condition ensures the stability and reliability of the numerical method. By confirming that the Hessian matrices are zero, we simplify the analysis and guarantee that the method will not encounter unbounded growth in errors. This property is crucial for the practical application of the method, as it assures that the numerical solutions will remain accurate and consistent over iterations.

3. NUMERICAL EXPERIMENTS

In this section, we study the numerical results of the implementation of the presented method on two examples.

Example 1. Let $T = 1, c_1 = 0, c_2 = 1, \tau_* = 0.1, \tau^* = 0.7, r = 0.05, \sigma = 0.5, \phi_0(x) = x(1 - x)$ and $\delta(x, t, \tau) = 0.01xt\tau^2$. The exact solution is assumed to be $\phi(x, t) = (1 + t)x(1 - x)$ and $g(x, t)$ can be obtained from exact solution.

The numerical and exact solutions for $N = 10, L = 10$, and $K = 20$ are shown in Figure 1. Also, the absolute errors in all mesh points are shown in Figure 2 for $N = 10, L = 10$, and $K = 20$.

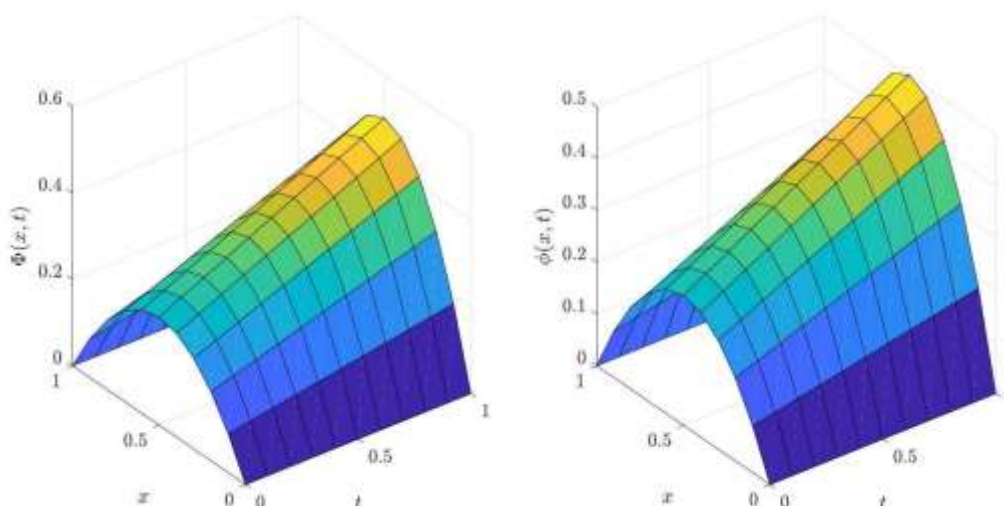


Figure 1: Numerical solution (left) and the exact solution (right) for all mesh points in Example 1 with $N = 10, L = 10$ and $K = 20$.

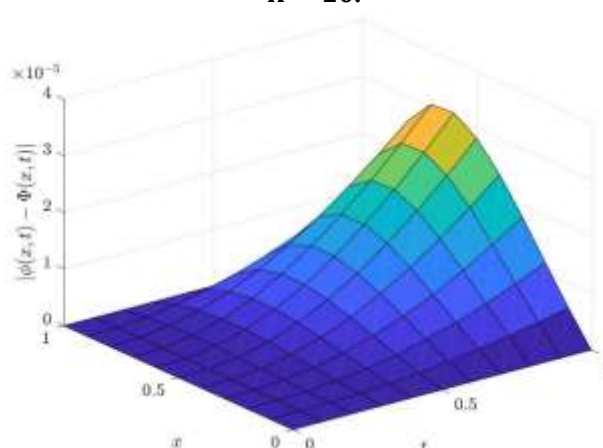


Figure 2: The absolute errors for all mesh points in Example 1 with $N = 10, L = 10$ and $K = 20$.

Example 2. Let $T = 1, c_1 = 0, c_2 = \pi, \tau_* = 0, \tau^* = 0.5, r = 0.01, \sigma = 0.2, \phi_0(x) = \sin(x), \delta(x, t, \tau) = 0.01\Gamma(1 - \tau)(x + 1)(t + 1)$. The exact solution is assumed to be $\phi(x, t) = (t^2 + 1)\sin(x)$ and $g(x, t)$ can be obtained from exact solution. The numerical and exact solutions for $N = 20, L = 20$, and $K = 30$ are shown in Figure 3. Also, the absolute errors in all mesh points are shown in Figure 4 for $N = 20, L = 20$, and $K = 30$.

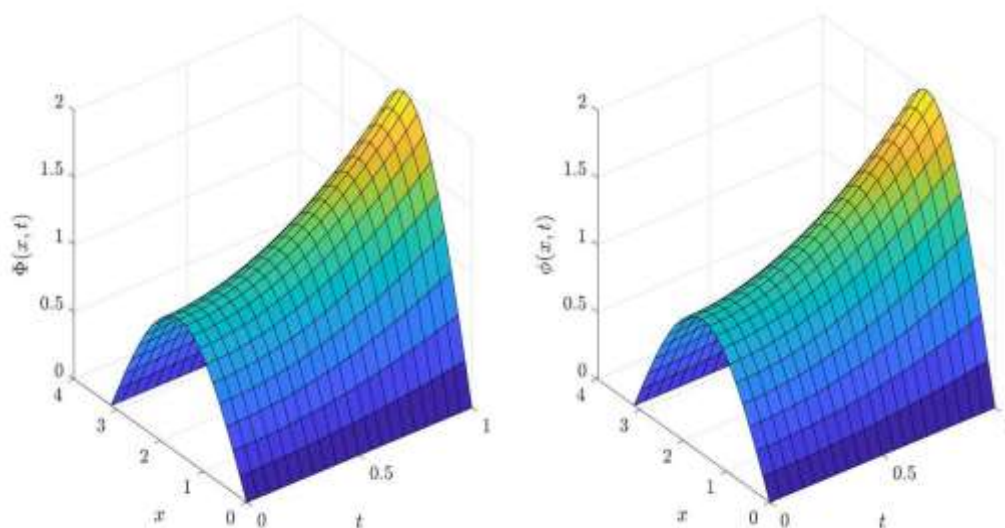


Figure 3: Numerical solution (left) and the exact solution (right) for all mesh points in Example 2 with $N = 20$, $L = 20$ and $K = 30$.

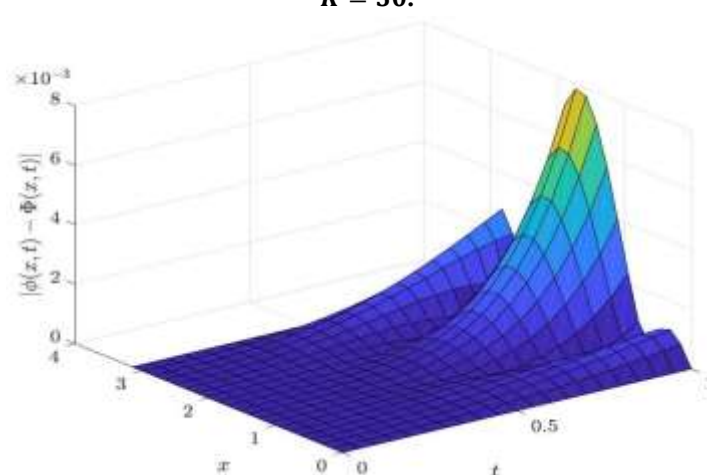


Figure 4: The absolute errors for all mesh points in Example 2 with $N = 20$, $L = 20$ and $K = 30$.

4. CONCLUSION

In conclusion, this research effectively addresses the nonlinear generalized distributed-order time-fractional Black-Scholes equation using an implicit method. By applying finite difference techniques to approximate both time and spatial derivatives, we have optimized the computational process, enhancing its efficiency. The numerical results confirm the accuracy and convergence rate of our method, demonstrating its robustness in solving financial models.

This study underscores the significant potential of finite difference methods in tackling complex mathematical equations within financial engineering. Moreover, our findings advance the broader understanding of numerical methods in financial mathematics, providing a dependable tool for both practitioners and researchers. The implications of this work extend beyond theoretical interest, offering practical solutions for real-world financial challenges. Future research could investigate the application of this method to other complex financial equations, potentially improving the accuracy and efficiency of financial modeling and risk assessment.

ACKNOWLEDGEMENTS

We are very grateful to anonymous referees for their careful reading and valuable comments which led to the improvement of this paper.



REFERENCES

1. F. Black, M. Scholes, The pricing of options and corporate liabilities, In World Scientific Reference on Contingent Claims Analysis in Corporate Finance, 1 (2019), 3-21.
2. R. C. Merton, Theory of rational option pricing, The Bell Journal of economics and management science, 4 (1973), 141 - 183.
3. HP. Deutsch, M.W. Beinker, The Black-Scholes Differential Equation. In: Derivatives and Internal Models. Finance and Capital Markets Series. Palgrave Macmillan, Cham. (2019), <https://doi.org/10.1007/978-3-030-22899-6-7>.
4. N. Burgess, A Review of the Generalized Black-Scholes Formula & Its Application to Different Underlying Assets, (2017). Available at SSRN: <https://ssrn.com/abstract=3023440> or <http://dx.doi.org/10.2139/ssrn.3023440>
5. S. C. S. Rao, Numerical solution of generalized Black Scholes model, Appl. Math. compute., 321 (2018), 401-421.
6. L. Anger Mann, S. Wang, Convergence of a fitted finite volume method for the penalized Black-Scholes equation governing European and American Option pricing, Numer. Math., 106 (2007), 1-40.
7. M. K. Kadalbajoo, L. P. Tripathi, A. Kumar, A cubic B-spline collocation method for a numerical solution of the generalized Black-Scholes equation, Math. Compute. Modelling, 55 (2012), 14831505.
8. H. Aminikhah, S. J. Alavi A new approach to using the cubic B-spline functions to solve the Black Scholes equation, Journal of New Researches in Mathematics. 5 (2019), 71-80.
9. D. Prathumwan, K. Trachoo, On the solution of two-dimensional fractional Black-Scholes equation for European put option, Adv. Difference Equ., 2020 (2020), 1-9.
10. H. Zhang, M. Zhang, F. Liu, M. Shen, Review of the Fractional Black-Scholes Equations and Their Solution Techniques. Fractal Fract., 8 (2024), 101. <https://doi.org/10.3390/fractalfract8020101>.
11. S.R. Saratha, G. Sai Sundara Krishnan, M. Bagyalakshmi, et al., Solving Black Scholes equations using fractional generalized homotopy analysis method. Comp. Appl. Math. 39, (2020), 262, <https://doi.org/10.1007/s40314-020-01306-4>.
12. A. Farhadi, M. Salehi, G.H. Erjaee, A New Version of Black Scholes Equation Presented by Time-Fractional Derivative, Iran J Sci Technol Trans Sci 42, (2018), 2159-2166, <https://doi.org/10.1007/s40995-017-0244-7>.
13. M. Zhang, J. Jia, X. Zheng, Numerical approximation and fast implementation to a generalized distributed-order time-fractional option pricing model, Chaos, Solitons and Fractals . 170 (2023), 113353.
14. J. Oliva-Maza, M. Warma, Introducing and solving generalized Black-Scholes PDEs through the use of functional calculus, J. Evol. Equ. 23 (2023), 10.
15. J. Alavi, H. Aminikhah, A Numerical Algorithm Based on Modified Orthogonal Linear Spline for Solving a Coupled Nonlinear Inverse Reaction Diffusion Problem, Filomat, 35, (2021) 79-104.
16. Y. X. Yuan, Review of trust region algorithms for optimization, ICIAM99 In: Proceedings of the Fourth International Congress on Industrial and Applied Mathematics. Oxford University Press, Edinburgh (2000).

Cite this Article: Ahmadi, N., Sayedi, S.L., Ahmadzai, H.G. (2025). A Dependable Numerical Approach for Solving a Nonlinear Generalized Fractional Distributed-Order Black-Scholes Equation. International Journal of Current Science Research and Review, 8(9), pp. 4683-4689. DOI: <https://doi.org/10.47191/ijcsrr/V8-i9-31>