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A Modish Glance of Integer Solutions to Non-homogeneous Cubic Diophantine Equation with Three Unknowns $7\,(x^2 - x\,y + y^2) = 12\,z^3$

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ABSTRACT: The main thrust of this paper is focused on determining varieties of non-zero distinct integer solutions to the non-homogeneous ternary cubic Diophantine equation $7\,(x^2 - x\,y + y^2) = 12\,z^3$. Various choices of integer solutions to the above equation are obtained by reducing it to the equation which is solvable, through employing suitable transformations and applying the factorization method.

KEYWORDS: ternary cubic, non-homogeneous cubic, integer solutions

INTRODUCTION

One of the interesting areas of Number Theory is the subject of Diophantine equations which has fascinated and motivated both Amateurs and Mathematicians alike. It is well-known that Diophantine equation is a polynomial equation in two or more unknowns requiring only integer solutions. It is quite obvious that Diophantine equations are rich in variety playing a significant role in the development of Mathematics. The theory of Diophantine equations is popular in recent years providing a fertile ground for both Professionals and Amateurs. In addition to known results, this abounds with unsolved problems. Although many of its results can be stated in simple and elegant terms, their proofs are sometimes long and complicated. There is no well unified body of knowledge concerning general methods. A Diophantine problem is considered as solved if a method is available to decide whether the problem is solvable or not and in case of its solvability , to exhibit all integers satisfying the requirements set forth in the problem. The successful completion of exhibiting all integers satisfying the requirements set forth in the problem add to further progress of Number Theory as they offer good applications in the field of Graph theory, Modular theory ,Coding and Cryptography ,Engineering ,Music and so on. Integers have repeatedly played a crucial role in the evolution of the Natural Sciences. The theory of integers provides answers to real world problems.

It is well-known that Diophantine equations, homogeneous or non-homogeneous, have aroused the interest of many mathematicians. It is worth to observe that Cubic Diophantine equations fall in to the theory of Elliptic curves which are used in Cryptography. In particular, one may refer [1-10] for cubic equations with three and four unknowns.

The main thrust of this paper is to exhibit different sets of integer solutions to an interesting ternary non-homogeneous cubic equation given by $7\,(x^2 - x\,y + y^2) = 12\,z^3$ by using elementary algebraic methods. The outstanding results in this study of Diophantine equation will be useful for all readers.

METHOD OF ANALYSIS

The non-homogeneous ternary cubic Diophantine equation to be solved is given by

$$7\,(x^2 - x\,y + y^2) = 12\,z^3 \qquad (1)$$

Different approaches of obtaining distinct integer solutions to (1) are exhibited below.

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Approach I

The substitution

$$x = k\,y, k \neq 1 \tag{2}$$

in (1) leads to

$$7(k^2 - k + 1)\; y^2 = 12\,z^3 \tag{3}$$

It is observed that (3) is satisfied by

$$y = 7*12^2*(k^2 - k + 1)t^{3s}, z = 7*12*(k^2 - k + 1)t^{2s} \tag{4}$$

Using (4) in (2), we get

$$x = 7*12^2*k\,(k^2 - k + 1)t^{3s} \tag{5}$$

Thus3, (4) & (5) represent the integer solutions to (1).

Approach 2

Taking

$$x = (k+1)v, y = (k-1)v\;, k \neq -1,1 \tag{6}$$

in (1), it leads to

$$7(k^2 + 3)v^2 = 12\,z^3 \tag{7}$$

which is satisfied by

$$v = 7*12^2*(k^2 + 3)t^{3s} \tag{8}$$

$$z = 7*12*(k^2 + 3)t^{2s} \tag{9}$$

Using (8) in (6), we have

$$\begin{aligned} x &= 7*12^2*(k+1)(k^2+3)t^{3s}, \\ y &= 7*12^2*(k-1)(k^2+3)t^{3s} \end{aligned} \tag{10}$$

Thus, (9), (10) represent the integer solutions to (1).

Approach 3

Taking

$$x = (1+k)u, y = (1-k)u\;, k \neq -1,1 \tag{11}$$

in (1), it leads to

$$7(1+3k^2)u^2 = 12\,z^3 \tag{12}$$

which is satisfied by

$$u = 7*12^2*(1+3k^2)t^{3s} \tag{13}$$

$$z = 7*12*(1+3k^2)t^{2s} \tag{14}$$

Using (13) in (11), we have

$$\begin{aligned} x &= 7*12^2*(1+k)(1+3k^2)t^{3s}, \\ y &= 7*12^2*(1-k)(1+3k^2)t^{3s} \end{aligned} \tag{15}$$

Thus, (14) ,(15) represent the integer solutions to (1).

Approach 4

The substitution

$$x = u + v, y = u - v, u \neq v \tag{16}$$

in (1) leads to

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$$7(u^2+3v^2)=12z^3 \tag{17}$$

Assume

$$z=a^2+3b^2 \tag{18}$$

Express the integers 7 & 12 as the product of complex conjugates as below:

$$7=(2+i\sqrt{3})(2-i\sqrt{3}) \tag{19}$$

and

$$12=(3+i\sqrt{3})(3-i\sqrt{3}) \tag{20}$$

Substituting (18) ,(19) and (20) in (17) and employing the method of factorization, consider

$$(2+i\sqrt{3})(u+i\sqrt{3}v)=(3+i\sqrt{3})(a+i\sqrt{3}b)^3 \tag{21}$$

Equating the real and imaginary parts in (21) and solving for u , v we have

$$\begin{aligned}7u&=9(a^3-9ab^2)+3(3a^2b-3b^3),\\7v&=-(a^3-9ab^2)+9(3a^2b-3b^3)\end{aligned} \tag{22}$$

As the interest is on finding integer solutions , replacing a by $7A$ & b by $7B$ in (22) & (18) and employing (16) ,the corresponding integer solutions to (1) are given by

$$\begin{aligned}x&=7^2[8(A^3-9AB^2)+12(3A^2B-3B^3)],\\y&=7^2[10(A^3-3AB^2)-6(3A^2B-3B^3)],\\z&=7^2(A^2+3B^2)\end{aligned}$$

Note: 1

It is worth mentioning that the integers 7 & 12 may also be written as

$$7=(2+i\sqrt{3})\ (2-i\sqrt{3})\ ,12=(i2\sqrt{3})(-i2\sqrt{3})$$

$$7=\frac{(1+i3\sqrt{3})(1-i3\sqrt{3})}{4}\ ,12=(3+i\sqrt{3})\ (3-i\sqrt{3}),$$

$$7=\frac{(1+i3\sqrt{3})(1-i3\sqrt{3})}{4}\ ,12=(i2\sqrt{3})\ (-i2\sqrt{3})$$

The repetition of the above process leads to different sets of integer solutions to (1).

Approach 5

Introduction of the transformations

$$v=14(3\alpha^2+1),z=7(3\alpha^2+1) \tag{23}$$

in (17) leads to

$$u=42\alpha(3\alpha^2+1) \tag{24}$$

In view of (16) , the corresponding values of x , y satisfying (1) are given by

$$\begin{aligned}x&=14(3\alpha+1)(3\alpha^2+1),\\y&=14(3\alpha-1)(3\alpha^2+1)\end{aligned}$$

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Approach 6

Introduction of the transformations

$$u = 42\ (s^2 + 3), z = 7(s^2 + 3) \quad (25)$$

in (17) leads to

$$v = 14\, s\,(s^2 + 3) \quad (26)$$

In view of (16) , the corresponding values of x , y satisfying (1) are given by

$$x = 14(3 + s)(s^2 + 3),$$
$$y = 14(3 - s)(s^2 + 3)$$

Approach 7

Taking

$$y = 28\beta, z = 7R \quad (27)$$

in (1) and treating (1) as a quadratic in x & solving for x , we have

$$x = 14[\beta \pm \sqrt{3(R^3 - \beta^2)}] \quad (28)$$

The square-root on the R.H.S. of (28) is removed when

$$R = a^2 + 3b^2, \beta = a^3 - 9ab^2 \quad (29)$$

Substituting (29) in (28) & (27) ,the two sets of integer solutions to (1) are given by

$$x = 14(a^3 - 9ab^2 + 9a^2b - 9b^3), y = 28(a^3 - 9ab^2), z = 7(a^2 + 3b^2),$$
$$x = 14(a^3 - 9ab^2 - 9a^2b + 9b^3), y = 28(a^3 - 9ab^2), z = 7(a^2 + 3b^2)$$

Note : 2

It is to be noted that the square-root on the R.H.S. of (28) is also removed when

$$R = m^2 + 3n^2, \beta = m^3 + 3mn^2$$

For this choice ,the corresponding two sets of integer solutions to (1) are represented by

$$x = 14[m(m^2 + 3n^2) \pm 3n(m^2 + 3n^2)], y = 28m(m^2 + 3n^2), z = 7(m^2 + 3n^2)$$

CONCLUSION

In this paper, we have made an attempt to find infinitely many non-zero distinct integer solutions to the non-homogeneous cubic equation with three unknowns given by $7(x^2 - xy + y^2) = 12z^3$. To conclude, one may search for other choices of solutions to the considered cubic equation with three unknowns and higher degree diophantine equations with multiple variables.

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